

# Combining local and non-local terms in a nonlinear elliptic problem<sup>1</sup>

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## Abstract

In this paper we study the existence, uniqueness, multiplicity and stability of positive solution of a non-linear elliptic problem that combines local and non-local terms taking the form of an integral in space. The proofs are mainly based on fixed point theorems, bifurcation techniques, sub-supersolutions and continuation arguments.

*AMS Classification.* ??.

*Keywords.* Non-local elliptic equations, bifurcation techniques, a priori bounds.

## 1 Introduction

Throughout this work we consider the following problem

$$\begin{cases} -\Delta u = \lambda u^p + \int_{\Omega} u^{\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded regular domain,  $\lambda \in \mathbb{R}$  and  $p, \beta > 0$ .

During recent years the so called non-local elliptic problems have attracted the attention of a lot of researchers due two main aspects: Firstly due to their mathematical importance. The presence of non-local terms provokes some difficulties which, sometimes, do not appear in the local ones. So, the behaviour of these problems may be, in general, distinct of their local counterpart. Secondly, these problems arise from practical motivations from Biology, Physics, Heat Transfer, Mechanics and so on, which makes their studies particularly interesting. See, for instance, the review paper [8].

In particular, in problem (1.1) there exists a combination of a local and a non-local terms in additive way. Observe that while for  $\lambda = 0$  equation (1.1) is a non-local elliptic equation, when  $\lambda < 0$  there is a competition between both terms. It is interesting to study

<sup>1</sup>AS have been supported by the Spanish Ministry of Science and Technology under Grant MTM2009-12367.

the behaviour of the set of positive solutions of (1.1) depending of the size of  $p$  and  $\beta$  and of course of the sign of  $\lambda$ .

Problem (1.1) has been previously analyzed in [14] and [12], at least to our knowledge, only the case  $\lambda \leq 0$ ,  $\beta > 1$  and  $p \geq 1$ . In both works, the parabolic problem related to (1.1) was studied. In particular, both works showed the value  $p = \beta$  represents a critical blow-up exponent. Indeed, they proved that if  $\beta > p$  or  $\beta = p$  and  $\lambda > -|\Omega|$ , the blow-up can occur in finite time. However, when  $\beta < p$  or  $\beta = p$  and  $\lambda \leq -|\Omega|$  all the solutions are global and bounded.

With respect to the elliptic problem (1.1), the authors proved the existence of positive solution for  $\lambda$  small in the particular case  $\lambda < 0$ ,  $p > \beta > 1$ . In this paper, we complete this study, and give results for all the values of  $p$  and  $\beta$ .

Before proceeding to the statement of the main results, we need to introduce some notation. Given regular, non-negative and non-trivial functions  $a, b$  and  $m$ , we denote by  $\lambda_1(-\Delta + m; a, b)$  the principal eigenvalue of the following integro-differential eigenvalue problem

$$-\Delta u + m(x)u - a(x) \int_{\Omega} b(x)u = \lambda u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad (1.2)$$

(see Section 2 for a detailed study of this problem). Denote also

$$\lambda_1 := \lambda_1(-\Delta; 0, 0) \quad \text{and} \quad \sigma_1 := \lambda_1(-\Delta; 1, 1).$$

We use the principal eigenvalues of (1.2) to characterize the stability of the solutions with respect to the parabolic counterpart problem. We say that a positive solution  $u_0$  of (1.1) is stable (resp. unstable) if the principal eigenvalue of the linearization of (1.1) around  $u_0$  is positive (resp. negative), i.e.,

$$\lambda_1(-\Delta - \lambda p u_0^{p-1}; \beta; u_0^{\beta-1}) > 0 \quad (\text{resp. } < 0.)$$

We also say that  $u_0$  is neutrally stable if it is zero. Observe that  $p$  and  $\beta$  can be less than one, and so the eigenvalue problem (1.2) can have singular terms.

We can now state our main results, which depend on the size of  $p$  and  $\beta$ .

First, it is clear that if  $(p, \beta) = (1, 1)$  then (1.1) is an eigenvalue problem and it possesses positive solution if  $\lambda = \sigma_1$ . So, we assume that  $(p, \beta) \neq (1, 1)$ .

In the case  $p = 1$  we can obtain:

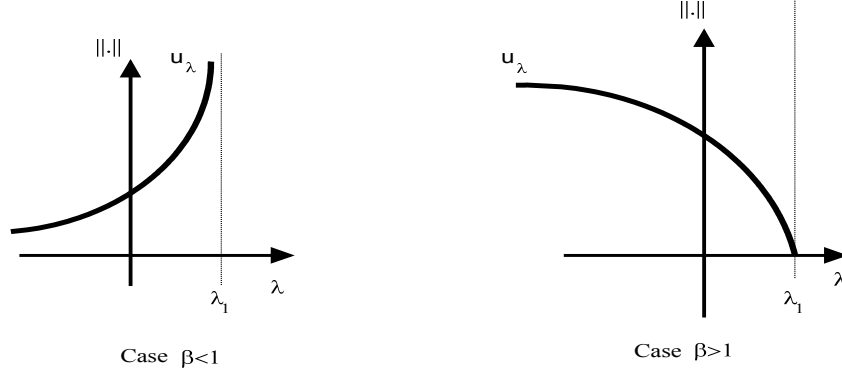
**Theorem 1.1.** *Assume that  $p = 1$ . Then, there exists a unique positive solution of (1.1) for  $\lambda < \lambda_1$ , and no positive solutions for  $\lambda \geq \lambda_1$ . Moreover, the solution is stable for  $\beta < 1$  and unstable for  $\beta > 1$ . Finally,*

$$\lim_{\lambda \rightarrow \lambda_1} \|u\|_{\infty} = \begin{cases} 0 & \text{when } \beta > 1, \\ +\infty & \text{when } \beta < 1. \end{cases} \quad \text{and} \quad \lim_{\lambda \rightarrow -\infty} \|u\|_{\infty} = \begin{cases} +\infty & \text{when } \beta > 1, \\ 0 & \text{when } \beta < 1. \end{cases}$$

In Figure 1 we have represented the bifurcation diagrams corresponding to the case  $p = 1$ . Case 1 represents the solutions of (1.1) when  $\beta < 1$  and Case 2 shows the case  $\beta > 1$ . Observe that we have a bifurcation from zero when  $\beta > 1$  and a bifurcation from infinity when  $\beta < 1$  at  $\lambda = \lambda_1$ .

In the case  $p < 1$ , we get:

**Theorem 1.2.** *Assume that  $p < 1$ .*

Figure 1: Bifurcation diagrams for equation (1.1) for  $p = 1$ .

a) Assume also that  $\beta = 1$ .

(a) If  $\sigma_1 > 0$  there exists a positive solution of (1.1) if, and only if,  $\lambda > 0$ . The solution is unique and stable. Moreover,

$$\lim_{\lambda \rightarrow 0} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = \infty.$$

(b) If  $\sigma_1 = 0$  there exists a positive solution of (1.1) if, and only if,  $\lambda = 0$ . There are infinite positive solutions and they are neutrally stable.

(c) If  $\sigma_1 < 0$  there exists a positive solution of (1.1) if, and only if,  $\lambda < 0$ . The solution is unique and unstable. Moreover,

$$\lim_{\lambda \rightarrow 0} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \rightarrow -\infty} \|u_\lambda\|_\infty = \infty.$$

b) Assume also that  $\beta > 1$ . There exists a value  $\bar{\lambda} > 0$  such that there exists a positive solution of (1.1) if and only if  $\lambda \leq \bar{\lambda}$ . There is a unique and unstable positive solution for  $\lambda \leq 0$  and at least two solutions,  $u_1^\lambda < u_2^\lambda$ , for  $\lambda > 0$  and small,  $u_1^\lambda$  is stable and  $u_2^\lambda$  unstable. Moreover,

$$\lim_{\lambda \rightarrow -\infty} \|u_\lambda\|_\infty = \infty \quad \text{and} \quad \lim_{\lambda \rightarrow 0} \|u_1^\lambda\|_\infty = 0.$$

c) Assume now that  $\beta < 1$ .

(a) If  $\beta < p$  there exists a positive solution of (1.1) for all  $\lambda \in \mathbb{R}$ . The solution is unique and stable. Moreover,

$$\lim_{\lambda \rightarrow -\infty} \|u_\lambda\|_\infty = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = \infty.$$

(b) If  $\beta = p$  there exists  $\lambda_0 < 0$  such that there exists positive solution if and only if  $\lambda > \lambda_0$ . In fact,  $\lambda_0 \in (-|\Omega|, -\int_\Omega \varphi_1^p)$ , being  $\varphi_1 > 0$  the eigenfunction associated to  $\lambda_1$  such that  $\|\varphi_1\|_\infty = 1$ . Furthermore, the solution is unique and stable and

$$\lim_{\lambda \downarrow \lambda_0} \|u_\lambda\|_\infty = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = \infty.$$

(c) If  $p < \beta$  there exists  $\lambda_0 < 0$  such that there exists positive solution if and only if  $\lambda \geq \lambda_0$ . Moreover, for  $\lambda \geq 0$  the solution is unique and stable, and for  $\lambda$  negative and small there exist at least two positive solutions,  $u_1^\lambda < u_2^\lambda$ ,  $u_1^\lambda$  is unstable and  $u_2^\lambda$  stable. Moreover,

$$\lim_{\lambda \rightarrow 0} \|u_2^\lambda\|_\infty = 0 \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = \infty.$$

In Figure 2 we have drawn the bifurcation diagrams of (1.1) corresponding to the case  $p < 1$ . Cases 1, 2 and 3 represent the solutions when  $\beta = 1$  and  $\sigma_1 > 0$ ,  $\sigma_1 = 0$  and  $\sigma_1 < 0$ , respectively. Case 4 shows the case  $\beta > 1$ , and when  $\beta < 1$  we have the Cases 5, 6 and 7 when  $\beta > p$ ,  $\beta = p$  and  $\beta < p$ , respectively.

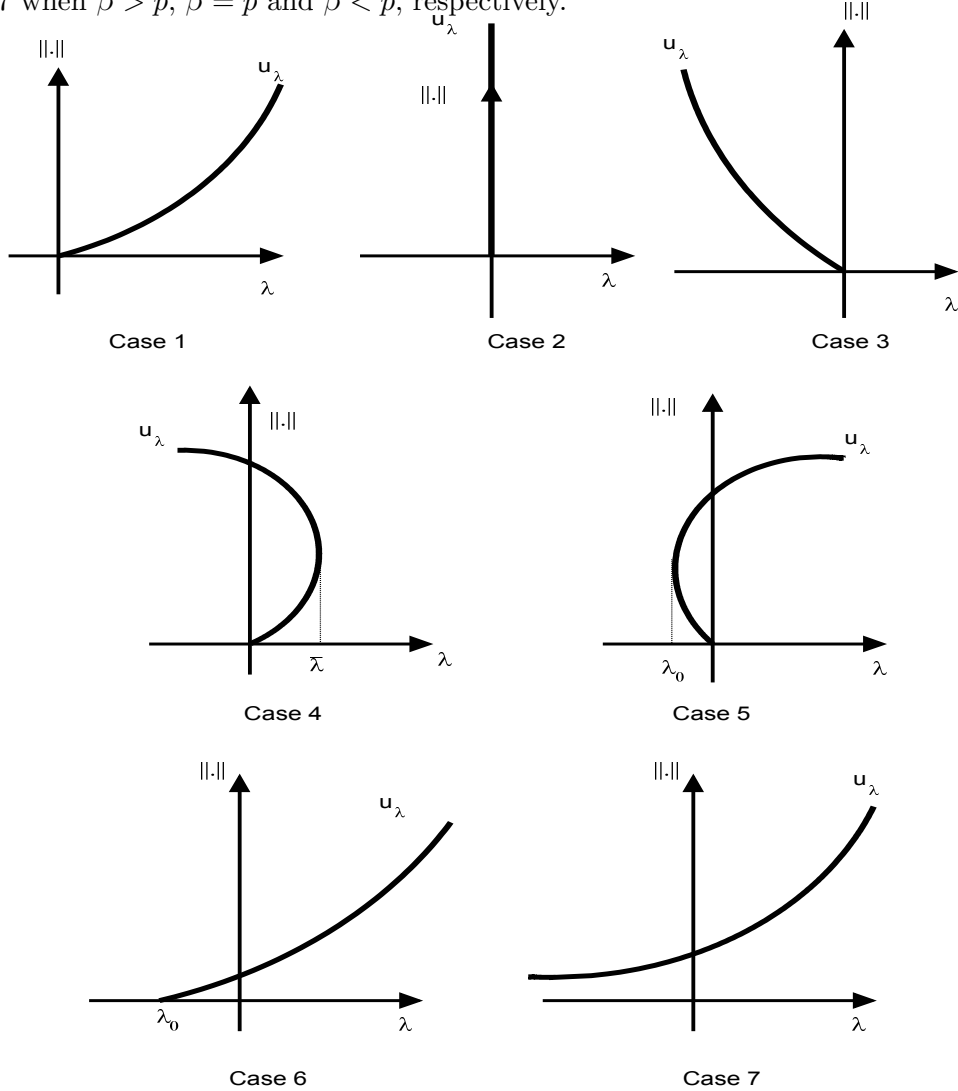


Figure 2: Bifurcation diagrams for equation (1.1) for  $p < 1$ .

Let us compare some of our results with the well-known ones of the local equation

$$-\Delta u = \lambda u^p + u^\beta.$$

In the case  $p = 1$  the existence results are rather similar to the local case. However, for the case  $\beta > 1$  in the non-local case we do not need impose the condition  $\beta < (N + 2)/(N - 2)$

to obtain the existence of a priori bounds. Moreover, in this case we show that the solution is unstable (similar to the local case) but the solution is unique, unlike the local case.

With respect to the case  $p < 1$  we would like to point out that in the non-local case any non-negative and non-trivial solution is positive in all  $\Omega$ . This contrasts with the local case in which for  $\lambda$  negative could exist non-negative and non-trivial solutions that vanishes in a part of  $\Omega$ , the dead core.

Observe that in the case  $p < 1 < \beta$  the result obtained is rather similar to the case of the local equation studied in [1]. However, again in our case we do not need to impose the condition  $\beta < (N + 2)/(N - 2)$ . Also, the result obtained in the case  $p < \beta < 1$  is similar to the local equation analyzed in [7].

Let us remark that to obtain the existence results in the previous results, we can not use the variational methods due to the equation (1.1) has not a variational structure. In fact, we have used basically a fixed point argument and the sub-supersolution method to obtain above results.

For the case  $p > 1$  we are not able to use the fixed point argument. So, we have introduced our equation (1.1) in a more general equation, see equation (4.18), and use bifurcation methods and classical results from [2]. For that, we need to obtain a priori bounds of positive solutions of (1.1). This is not a trivial problem. We distinguish two cases. When  $\lambda < 0$  we obtain a priori bounds except in the case  $\beta = p \geq 1$ . The case  $\lambda > 0$  is harder. Basically, we have used two different arguments: boot-strapping and blow-up arguments to obtain the results. For the case  $\lambda > 0$  we have proved that if

$$p < 1, \forall \beta > 0 \quad \text{or} \quad p = 1, \beta > 1, \quad (1.3)$$

or,

$$1 < p < (N + 2)/(N - 2), \quad \forall \beta > 0, \quad (1.4)$$

or

$$p \geq (N + 2)/(N - 2), \quad \text{and} \quad \beta > (N/2)(p - 1), \quad (1.5)$$

then there exist a priori bounds of (1.1). Observe that (1.4) is the classical restriction in the local case. On the other hand, (1.3) means that when  $p$  is small, we obtain a priori bounds for all the values of  $\beta$ ; while (1.5) gives a priori bounds when  $\beta$  is large, even when  $p$  is greater than critical exponent  $(N + 2)/(N - 2)$ .

Moreover, these results are optimal in some way, because for  $\lambda$  negative and  $\beta = p > 1$  and for  $\lambda$  positive and  $\beta < 1 = p$  we prove that there exist a bifurcation from infinity for some  $\lambda$ , and so a priori bounds do not exist. Moreover, we show that for  $p = \beta > (N + 2)/(N - 2)$  there is not positive solution for  $\lambda$  large.

**Theorem 1.3.** *Assume that  $p > 1$ .*

*a) Assume that  $\beta = 1$ .*

*(a) Suppose that  $\sigma_1 > 0$ . If there exists a positive solution then  $\lambda > 0$ . If  $\lambda > 0$  and  $p < (N + 2)/(N - 2)$  then there exists at least a positive solution. The solution is unstable and*

$$\lim_{\lambda \rightarrow 0} \|u_\lambda\|_\infty = \infty \quad \text{and} \quad \lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = 0. \quad (1.6)$$

*(b) If  $\sigma_1 = 0$  there exists a positive solution if, and only if,  $\lambda = 0$ . There are infinite positive solutions and they are neutrally stable.*

(c) If  $\sigma_1 < 0$  there exists a positive solution if, and only if,  $\lambda < 0$ . The solution is unique and stable and

$$\lim_{\lambda \rightarrow -\infty} \|u_\lambda\|_\infty = 0, \quad \lim_{\lambda \rightarrow 0} \|u_\lambda\|_\infty = \infty. \quad (1.7)$$

b) Assume also that  $\beta > 1$ .

(a) If  $\beta > p$  there exists a unique and unstable positive solution for  $\lambda \leq 0$ . If, moreover, there exist a priori bounds, there exists positive solutions for  $\lambda > 0$  and it is unstable. Moreover,

$$\lim_{\lambda \rightarrow -\infty} \|u_\lambda\|_\infty = \infty.$$

(b) If  $\beta = p$ , there exists  $\lambda_0 < 0$  such that (1.1) possesses positive solution for  $\lambda \in (\lambda_0, 0]$  and

$$\lim_{\lambda \rightarrow \lambda_0} \|u\|_\infty = +\infty. \quad (1.8)$$

Moreover, this solution is unique and unstable.

If, moreover, there exist a priori bounds, there exists positive solutions for  $\lambda > 0$  and it is unstable.

(c) If  $\beta < p$ , there exists  $\lambda_0 < 0$  such that (1.1) possesses positive solution for  $\lambda \in [\lambda_0, 0]$ . Moreover, if  $\lambda$  is small and negative, there exist at least two positive solutions,  $u_1^\lambda < u_2^\lambda$ ,  $u_1^\lambda$  is unstable and  $u_2^\lambda$  stable and

$$\lim_{\lambda \uparrow 0} \|u_2^\lambda\|_\infty = +\infty.$$

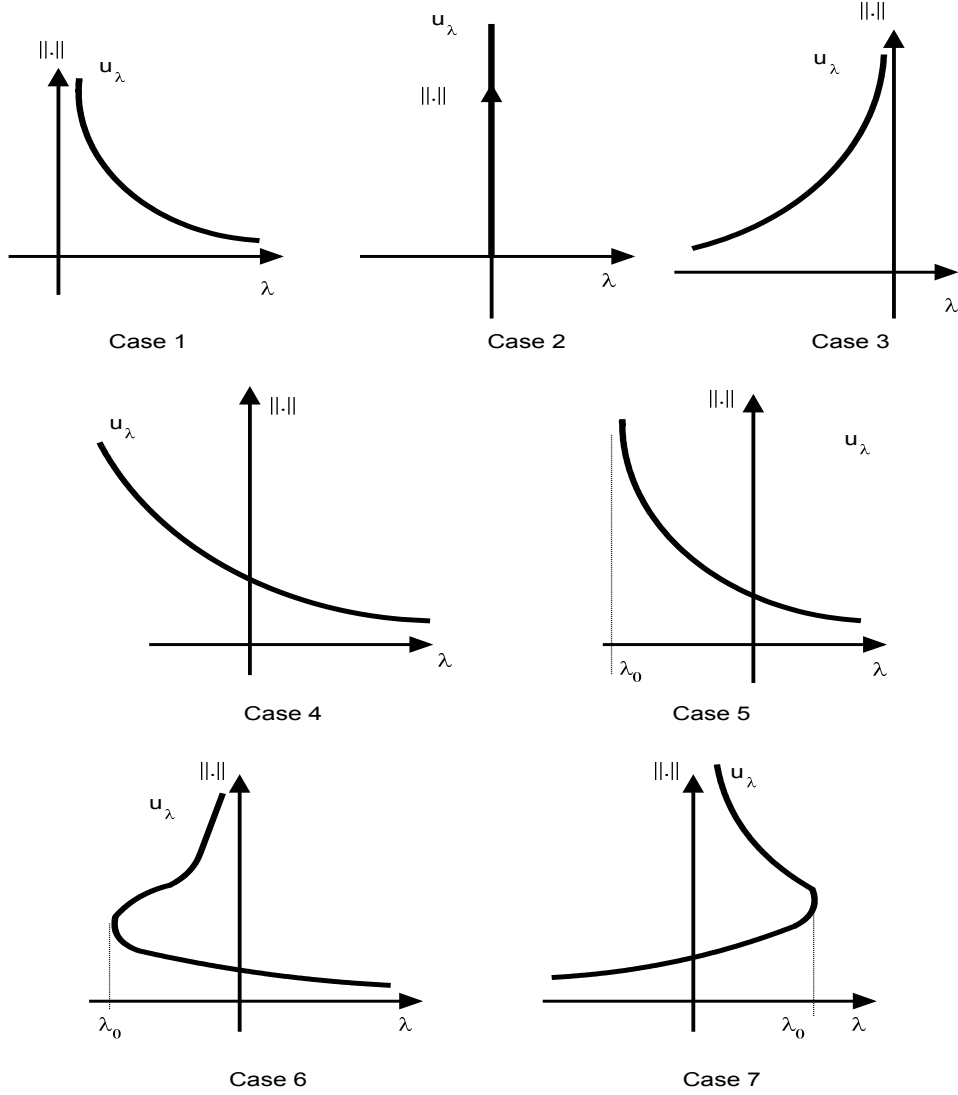
If, moreover, there exist a priori bounds, there exists positive solutions for  $\lambda > 0$  and it is unstable.

c) Assume now that  $\beta < 1$ . There exists a unique and stable positive solution of (1.1) for  $\lambda \leq 0$ . Assume now the existence of a a priori bounds. There exists  $\bar{\lambda} > 0$  such that there exists a positive solution if, and only if,  $\lambda \leq \bar{\lambda}$ . Moreover,  $\lambda > 0$  and small there exist at least two positive solutions  $u_1^\lambda, u_2^\lambda$ ,  $u_1^\lambda$  is stable and

$$\lim_{\lambda \rightarrow -\infty} \|u_\lambda\|_\infty = +0, \quad \lim_{\lambda \uparrow 0} \|u_2^\lambda\|_\infty = +\infty.$$

In Figure 3 we have represented the bifurcation diagrams of (1.1) corresponding to the case  $p > 1$ . Cases 1, 2 and 3 represent the solutions when  $\beta = 1$  and  $\sigma_1 > 0$ ,  $\sigma_1 = 0$  and  $\sigma_1 < 0$ , respectively. Cases 4, 5 and 6 show the cases  $\beta > p$ ,  $\beta = p$  and  $\beta < p$ , respectively. Finally, Case 7 represents  $\beta < 1$ .

An outline of the paper is: in Section 2 we study the eigenvalue problem and some preliminaries results; Section 3 is devoted to obtain a priori bounds of positive solutions of (1.1) and in the last Section we prove Theorems 1.1, 1.2 and 1.3.

Figure 3: Bifurcation diagrams for equation (1.1) for  $p > 1$ .

## 2 The eigenvalue problem and preliminaries results

In this section we study a non-local and singular eigenvalue problem, which appears when one linearizes around a positive solution of (1.1). Specifically, we study the following problem

$$\begin{cases} -\Delta u + m(x)u - a(x) \int_{\Omega} b(x)u = \sigma u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.1)$$

where  $m \in C^1(\Omega)$ ,  $a \in C(\overline{\Omega})$  and  $b \in C^1(\Omega)$  and verify: for some  $\alpha \in (-1, 1)$  and  $\gamma < 1$

(Hm)  $|\partial_i m| d(x, \partial\Omega)^{2-\alpha}$  are bounded for all  $x \in \Omega$  and  $i = 1, \dots, N$ ;

(Hb) there exists  $K > 0$  such that  $b(x) \leq K d(x, \partial\Omega)^{-\gamma}$ ,

where  $d(x, \partial\Omega) := \text{dist}(x, \partial\Omega)$ . The next result was proved in [4]:

**Theorem 2.1.** Assume that  $m$  verifies  $(Hm)$ ,  $a \in C^1(\Omega) \cap C(\overline{\Omega})$ , is a non-negative and non-trivial function,  $b \in C^1(\Omega)$  is a non-negative and non-trivial function and it verifies  $(Hb)$ . Then, there exists a principal eigenvalue of (2.1), denoted by  $\lambda_1(-\Delta + m; a; b)$ , which has an associated positive eigenfunction  $\varphi_1 \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$  for some  $\delta \in (0, 1)$ , and

$$\frac{\partial \varphi_1}{\partial n} < 0 \quad \text{on } \partial\Omega, \quad (2.2)$$

where  $n$  denotes the outward unit normal vector. Moreover,  $\lambda_1(-\Delta + m; a; b)$  is simple, and it is the unique eigenvalue having an associated eigenfunction without change of sign.

In the following result we give a criteria to ascertain the sign of  $\lambda_1(-\Delta + m; a; b)$ , see also [4]:

**Proposition 2.2.** a) Assume that there exists a positive function  $\bar{u} \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$ ,  $\delta \in (0, 1)$ , such that

$$-\Delta \bar{u} + m(x)\bar{u} - a(x) \int_{\Omega} b(x)\bar{u} > 0 \quad \text{in } \Omega.$$

Then,

$$\lambda_1(-\Delta + m; a; b) > 0.$$

b) Assume that there exists a positive function  $\underline{u} \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$ ,  $\delta \in (0, 1)$ , such that

$$-\Delta \underline{u} + m(x)\underline{u} - a(x) \int_{\Omega} b(x)\underline{u} < 0 \quad \text{in } \Omega.$$

Then,

$$\lambda_1(-\Delta + m; a; b) < 0.$$

Along the paper, we are going to denote by

$$\lambda_1 := \lambda_1(-\Delta; 0; 0) \quad \text{and} \quad \sigma_1 := \lambda_1(-\Delta; 1; 1).$$

The next result characterizes the sign of  $\lambda_1(-\Delta + m; a; b)$  on terms of the solution of the problem

$$\begin{cases} -\Delta \zeta + m(x)\zeta = b(x) & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.3)$$

Thanks to Proposition 2.5 in [11] if  $\lambda_1(-\Delta + m; 0; 0) > 0$  there exists a unique positive solution  $\zeta \in C^2(\Omega) \cap C_0^{1,\delta}(\overline{\Omega})$ ,  $\delta \in (0, 1)$ , of (2.3).

**Lemma 2.3.** Assume that  $\lambda_1(-\Delta + m; 0; 0) > 0$ . Then,

$$\text{sgn}(\lambda_1(-\Delta + m; a; b)) = \text{sgn}\left(1 - \int_{\Omega} a(x)\zeta\right).$$

*Proof.* Denote by  $\varphi_1$  a positive eigenfunction associated to  $\lambda_1(-\Delta + m; a; b)$ . Multiplying (2.3) by  $\varphi_1$ , and integrating we obtain that

$$\left(1 - \int_{\Omega} a(x)\zeta\right) \int_{\Omega} b(x)\varphi_1 = \lambda_1(-\Delta + m; a; b) \int_{\Omega} \varphi_1 \zeta.$$

This concludes the result. ■



The next result shows the monotony of the principal eigenvalue with respect to the domain.

**Lemma 2.4.** *Consider a sub-domain  $\Omega_0 \subset \Omega$ , and that the functions  $a, b$  and  $m$  verify the conditions of Theorem 2.1. Denote by  $\lambda_0$  and  $\lambda_1$  the principal eigenvalues  $\lambda_1(-\Delta + m; a; b)$  in  $\Omega_0$  and  $\Omega$ , respectively. Then,  $\lambda_1 < \lambda_0$ .*

*Proof.* Consider  $\varphi_0^*$  the adjoint positive eigenfunction associated to  $\lambda_0$ , that is

$$-\Delta \varphi_0^* + m(x) \varphi_0^* - b(x) \int_{\Omega_0} a(x) \varphi_0^* = \lambda_0 \varphi_0^* \quad \text{in } \Omega_0, \quad \varphi_0^* = 0 \quad \text{on } \partial \Omega_0.$$

Then, prolonging  $\varphi_0^*$  by zero at  $\Omega$ , and multiplying by  $\varphi_1$ , a positive eigenfunction associated to  $\lambda_1$ , we get

$$\int_{\Omega_0} a(x) \varphi_0^* \left[ \int_{\Omega_0} b(x) \varphi_1 - \int_{\Omega} b(x) \varphi_1 \right] + \int_{\partial \Omega_0} \frac{\partial \varphi_0^*}{\partial n} \varphi_1 = (\lambda_1 - \lambda_0) \int_{\Omega_0} \varphi_0^* \varphi_1,$$

whence, using (2.2), we deduce that  $\lambda_1 < \lambda_0$ . ■

With respect to the monotony on the potentials, we have:

**Lemma 2.5.** *Assume that  $m_1 \leq m_2$ ,  $a_1 \geq a_2$  and  $b_1 \geq b_2$ . Then,*

$$\lambda_1(-\Delta + m_1; a_1; b_1) \leq \lambda_1(-\Delta + m_2; a_2; b_2).$$

*Proof.* Let  $\varphi > 0$  an eigenfunction associated to  $\lambda_1(-\Delta + m_1; a_1; b_1)$ . Then

$$-\Delta \varphi + m_2 \varphi - \lambda_1(-\Delta + m_1; a_1; b_1) \varphi - a_2 \int_{\Omega} b_2 \varphi = (m_2 - m_1) \varphi + a_1 \int_{\Omega} b_1 \varphi - a_2 \int_{\Omega} b_2 \varphi \geq 0,$$

and so, by Proposition 2.2,  $\lambda_1(-\Delta + m_2 - \lambda_1(-\Delta + m_1; a_1; b_1); a_2; b_2) \geq 0$ , that is,

$$\lambda_1(-\Delta + m_2; a_2; b_2) \geq \lambda_1(-\Delta + m_1; a_1; b_1). \quad \text{■}$$

Finally, the following result will be very useful during the work:

**Lemma 2.6.** *Assume  $a > 0$  in  $\Omega$ . It holds that*

$$\lim_{\lambda \rightarrow +\infty} \lambda_1(-\Delta + m; \lambda a; b) = -\infty.$$

*Proof.* Consider a ball  $B \subset \Omega$  such that  $b \geq b_0 > 0$  in  $B$  and such that  $\lambda_1^B(-\Delta + m; 0; 0) > 0$ . By Lemma 2.4

$$\lambda_1^\Omega(-\Delta + m; \lambda a; b) < \lambda_1^B(-\Delta + m; \lambda a; b).$$

We are going to prove that  $\lambda_1^B(-\Delta + m; \lambda a; b) \rightarrow -\infty$  as  $\lambda \rightarrow +\infty$ . Indeed, since  $\lambda_1^B(-\Delta + m; 0; 0) > 0$  there exists a unique positive solution, denoted by  $e$ , of the equation

$$-\Delta e + m(x)e = b(x) \quad \text{in } B, \quad e = 0 \quad \text{on } \partial B.$$

Denote by  $\varphi > 0$  an eigenfunction associated to  $\lambda_1^B(-\Delta + m; \lambda a; b)$ . Multiplying the equation that verifies  $\varphi$  by  $e$  and integrating, we obtain

$$\lambda_1^B(-\Delta + m; \lambda a; b) = \left(1 - \lambda \int_B a e\right) \left[ \frac{\int_B b(x) \varphi}{\int_B \varphi e} \right]. \quad (2.4)$$

Observe that

$$\int_B \varphi e \leq \|e\|_\infty \int_B \varphi = \frac{\|e\|_\infty}{b_0} b_0 \int_B \varphi \leq C \int_B \varphi b(x),$$

and so

$$0 < \frac{1}{C} \leq \frac{\int_B b(x) \varphi}{\int_B \varphi e}.$$

It suffices to take  $\lambda \rightarrow -\infty$  in (2.4). ■

Now, we prove some results concerning to the problem (1.1). First, we have the following result about the positivity and bounds of the solutions of (1.1).

**Lemma 2.7.** *Assume that  $u$  is a non-negative and non-trivial solution of (1.1). Then,*

a)  *$u$  is strictly positive.*

b) *It holds*

$$-\lambda \|u\|_\infty^p \leq \int_\Omega u^\beta. \quad (2.5)$$

*Proof.* a) The result is evident if  $\lambda \geq 0$ . So, assume that  $\lambda < 0$  and fixed. Take a non-negative and non-trivial solution  $u$  of (1.1). Then, if  $p \geq 1$  it is clear that there exists  $M > 0$  such that  $\lambda u^p + Mu > 0$ . If  $p < 1$  then

$$Mu + \lambda u^p + \int_\Omega u^\beta = Mu + \lambda u^p + R_\lambda \geq M^{p/(p-1)} \left( \frac{1}{-\lambda} \right)^{1/(p-1)} p^{p/(1-p)} (p-1) + R_\lambda > 0$$

taking  $M$  large. Then, in both cases

$$-\Delta u + Mu > 0 \quad \text{in } \Omega,$$

and the result concludes using the strong maximum principle.

b) By the maximum principle we obtain (2.5). ■

With respect to the behaviour as  $\lambda \rightarrow -\infty$  and  $\lambda \rightarrow \infty$  we have:

**Lemma 2.8.** *In the case that the solution exists*

$$\lim_{\lambda \rightarrow -\infty} \|u_\lambda\|_\infty = \begin{cases} +\infty & \text{if } \beta > p, \\ 0 & \text{if } \beta < p, \end{cases} \quad \lim_{\lambda \rightarrow \infty} \|u_\lambda\|_\infty = \begin{cases} +\infty & \text{if } p < 1, \\ 0 & \text{if } 1 < p \text{ and } \beta = 1. \end{cases}$$

*Proof.* From (2.5) we get that

$$-\lambda \leq |\Omega| \|u_\lambda\|_\infty^{\beta-p}.$$

This concludes the first limit.

For the second one, observe that  $-\Delta u \geq \lambda u^p$  and then  $u$  is supersolution of the equation

$$-\Delta w = w^p \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega. \quad (2.6)$$

Denoting  $w_1$  the unique positive of (2.6) we arrive at

$$u_\lambda \geq \lambda^{1/(1-p)} w_1.$$

On the other hand, when  $\beta = 1$  and assuming that there exists positive solution for all  $\lambda > 0$ , we have that

$$u_\lambda = \lambda^{1/(1-p)} u_1, \quad (2.7)$$

where  $u_1$  is a positive solution of (1.1) when  $\lambda = 1$ . We finish the result.  $\blacksquare$

In the following result we prove the stability of a positive solution of (1.1).

**Proposition 2.9.** *Let  $u_0$  be a positive solution of (1.1). Then,*

- a) *If  $\beta \leq 1$ ,  $\lambda(1-p) \geq 0$  and some inequality strict,  $u_0$  is stable.*
- b) *If  $\beta \geq 1$ ,  $\lambda(1-p) \leq 0$  and some inequality strict,  $u_0$  is unstable.*
- c) *If  $\lambda < 0$ ,  $\beta \leq p < 1$ ,  $u_0$  is stable.*

*Proof.* We have to study the sign of the eigenvalue problem

$$\begin{cases} -\Delta \xi - \lambda p u_0^{p-1} \xi - \beta \int_\Omega u_0^{\beta-1} \xi = \sigma \xi & \text{in } \Omega, \\ \xi = 0 & \text{on } \partial\Omega. \end{cases}$$

First, observe that by Lemma 2.7  $u_0$  is strictly positive, so there exist positive constants  $C_i > 0$ ,  $i = 1, 2$  such that

$$0 < C_1 d(x, \partial\Omega) \leq u_0(x) \leq C_2 d(x, \partial\Omega),$$

and so the above eigenvalue problem is in the setting of (2.1).

Using Proposition 2.2 and taking now  $\bar{u} = u_0$  we obtain

$$-\Delta \bar{u} - \lambda p u_0^{p-1} \bar{u} - \beta \int_\Omega u_0^{\beta-1} \bar{u} = \lambda(1-p) u_0^p + (1-\beta) \int_\Omega u_0^\beta.$$

So, if  $\beta \leq 1$  and  $\lambda(1-p) \geq 0$  and some inequality strict,  $u_0$  is stable. Similarly in the second case.

For the third paragraph, observe that using (2.5) we get

$$-\lambda(1-p) u_0^p < -\lambda(1-p) \|u_0\|_\infty^p \leq (1-\beta) \int_\Omega u_0^\beta.$$

This concludes the result.  $\blacksquare$

### 3 A priori bounds

In this section we prove some results on a priori bounds of positive solutions of (1.1). Firstly, we denote by  $e$  the unique positive solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.1)$$

**Lemma 3.1.** *Let  $u$  be a positive solution of (1.1). Then if  $\lambda \geq 0$  (resp.  $\lambda \leq 0$ )*

$$u \geq e \int_{\Omega} u^{\beta} \quad (\text{resp.} \quad u \leq e \int_{\Omega} u^{\beta}).$$

*Proof.* Observe that if  $\lambda \geq 0$

$$-\Delta u = \lambda u^p + \int_{\Omega} u^{\beta} \geq \int_{\Omega} u^{\beta},$$

and so,

$$-\Delta \left( \frac{u}{\int_{\Omega} u^{\beta}} \right) \geq 1.$$

Whence we conclude that

$$u \geq e \int_{\Omega} u^{\beta}.$$

The case  $\lambda \leq 0$  is performed in a similar way. This completes the proof.  $\blacksquare$

In the next result, we show the existence of a priori bounds of positive solutions of (1.1) for  $\lambda$  negative.

**Proposition 3.2.** *Assume  $\lambda \in \Lambda \subset (-\infty, 0)$ ,  $\Lambda$  a compact,  $p, \beta > 0$  and*

$$(p, \beta) \notin \{(p, \beta) : p = \beta \geq 1\}.$$

*Then*

$$\|u\|_{\infty} \leq C, \quad \text{for some positive constant } C > 0 \text{ independent of } u.$$

*Proof.* Assume first that  $\beta < 1$ . Then, by Lemma 3.1 we get that

$$u \leq e \int_{\Omega} u^{\beta} \leq e \left( \int_{\Omega} e^{\beta} \right)^{1/(1-\beta)}.$$

On the other hand, observe that by (2.5) we have that

$$-\lambda \|u\|_{\infty}^{p-\beta} \leq |\Omega|,$$

and so the result follows for  $p > \beta$ .

Now, assume that  $\beta \geq 1$  and  $p \leq \beta$ . Suppose that there exists a sequence  $(\lambda_n, u_n)$ ,  $\lambda_n \in \Lambda$ ,  $\lambda_n \rightarrow \lambda_0 < 0$  and  $u_n$  positive solutions of (1.1) such that  $\|u_n\|_{\infty} \rightarrow \infty$ . Denote by

$$t_n := \int_{\Omega} u_n^{\beta},$$

then  $u_n$  is the unique positive solution of the equation

$$-\Delta u_n = \lambda_n u_n^p + t_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega. \quad (3.2)$$

Observe that since  $\lambda_n < 0$ , the map  $s \mapsto \lambda_n s^p + t_n$  is decreasing, and so the uniqueness follows. By the maximum principle, we obtain that

$$\lambda_n \|u_n\|_\infty^p + t_n \geq 0,$$

and so  $t_n \rightarrow \infty$ .

On the other hand, given  $\delta > 0$  for  $n \geq n_0 \in \mathbb{N}$  we get  $\lambda_0 - \delta < \lambda_n < \lambda_0 + \delta$ . Consider  $\varepsilon > 0$  and  $\varphi_1 > 0$  a positive eigenfunction associated to  $\lambda_1$  such that  $\|\varphi_1\|_\infty = 1$ , then  $\varepsilon\varphi_1$  is sub-solution of (3.2) for  $n \geq n_0$  if

$$\varepsilon\lambda_1 = (\lambda_0 - \delta)\varepsilon^p + t_n.$$

Hence, since  $t_n \rightarrow \infty$  we have that  $\varepsilon \rightarrow \infty$  and in fact,

$$\varepsilon = \frac{t_n}{\lambda_1 - (\lambda_0 - \delta)} \quad \text{if } p = 1, \quad \frac{\varepsilon^p}{t_n} \rightarrow \frac{1}{-(\lambda_0 - \delta)} \quad \text{if } p > 1, \quad \frac{\varepsilon}{t_n} \rightarrow \frac{1}{\lambda_1} \quad \text{if } p < 1.$$

Now, using that  $\varepsilon\varphi_1 \leq u_n$  we get that

$$\varepsilon^\beta \int_\Omega \varphi_1^\beta \leq t_n.$$

This is a contradiction for the cases  $1 = p < \beta$ ,  $1 < p < \beta$  and  $p < 1 < \beta$ .

Finally, we consider the case  $p < 1 = \beta$ . Observe first that it there exists a positive solution for  $\lambda_n < 0$ , then

$$-\Delta u_n - \int_\Omega u_n < 0 \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega,$$

and so by Proposition 2.2 it follows that  $\sigma_1 < 0$ . Denoting by

$$w_n = \frac{u_n}{\|u_n\|_\infty},$$

we have that

$$-\Delta w_n - \int_\Omega w_n = \lambda_n w_n^p \|u_n\|^{p-1},$$

and so, passing to the limit, we get that  $w_n \rightarrow w$  in  $C^2(\overline{\Omega})$  with

$$-\Delta w - \int_\Omega w = 0 \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

Since  $\sigma_1 < 0$  we conclude that  $w \equiv 0$ , a contradiction with  $\|w\|_\infty = 1$ . ■

Now we treat with the case  $\lambda$  positive.

**Proposition 3.3.** *Assume  $\lambda \geq 0$ ,  $\beta > 1$  and  $\beta > \max\{p, (N/2)(p-1)\}$  and  $u$  is a positive solution of (1.1). Then*

$$\|u\|_\infty \leq C, \quad \text{for some positive constant } C > 0 \text{ independent of } u.$$

*Proof.* We are going to use a boot-strapping argument. For that we set

$$f(u) := \lambda u^p + \int_{\Omega} u^{\beta}.$$

Thanks to Lemma 3.1, we have

$$\left( \int_{\Omega} u^{\beta} \right)^{\beta-1} \leq \left( \int_{\Omega} e^{\beta} \right)^{-1},$$

and then  $f(u)$  is bounded in  $L^{\beta/p}(\Omega)$  and which implies that  $u$  is bounded in  $W^{2,\beta/p}(\Omega)$ . Consequently, if  $\beta/p \geq N/2$ , then  $u$  is bounded in  $L^{\infty}(\Omega)$ . Assume that  $\beta/p < N/2$ . In this case,  $u$  is bounded in  $L^{\beta_1}(\Omega)$  with

$$\frac{1}{\beta_1} = \frac{p}{\beta} - \frac{2}{N} = \frac{pN - 2\beta}{\beta N}.$$

Then,  $f(u)$  is bounded in  $L^{\beta_1/p}(\Omega)$  and so  $u$  in  $W^{2,\beta_1/p}(\Omega)$ . Again, if

$$\frac{\beta_1}{p} \geq N/2,$$

we have the a priori bound. Assume that

$$\frac{\beta_1}{p} < \frac{N}{2}.$$

Now,  $u$  is bounded in  $L^{\beta_2}(\Omega)$  where

$$\frac{1}{\beta_2} = \frac{p}{\beta_1} - \frac{2}{N} = \frac{p^2 N - 2\beta(p+1)}{\beta N}$$

and then  $f(u)$  is bounded in  $L^{\beta_2/p}(\Omega)$ . Applying this reasoning  $n$  times, we have a priori bound if

$$\frac{\beta}{p} \geq \frac{N}{2} \frac{p^n}{p^n + p^{n-1} + \dots + p + 1}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{p^n}{p^n + p^{n-1} + \dots + p + 1} = \frac{p-1}{p},$$

we conclude the result. ■

Consider now the following equation

$$\begin{cases} -\Delta u = \lambda u^p + t & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.3)$$

where  $t > 0$ .

**Proposition 3.4.** *Consider  $\lambda > 0$ , a sequence  $t_n > 0$  and a sequence of positive solutions  $u_n$  of (3.3) such that  $\|u_n\|_{\infty} \rightarrow \infty$ . Then, if  $1 < p < (N+2)/(N-2)$ ,*

$$\|u_n\|_{\infty} \leq C t_n^{1/p} \quad \text{for some positive constant } C.$$

*Proof.* We are going to use a Gidas-Spruck argument, see [10]. Denote by

$$M_n := \|u_n\|_\infty$$

and  $x_n \in \Omega$  such that  $M_n = u_n(x_n)$ .

Assume that

$$M_n t_n^{-1/p} \rightarrow \infty. \quad (3.4)$$

Let

$$w_n(y) := \frac{u_n(M_n^{\frac{1-p}{2}} y + x_n)}{M_n},$$

defined in  $\Omega_n = \{y \in \mathbb{R}^N : x_n + M_n^{\frac{1-p}{2}} y \in \Omega\}$ .

Then, it is easy to show that  $w_n$  verifies

$$-\Delta w_n = \lambda w_n^p + (M_n t_n^{-1/p})^{-p} \quad \text{in } \Omega_n,$$

and  $0 \leq w_n \leq 1$ ,  $w_n(0) = 1$ .

Using the compactness of  $\overline{\Omega}$ , we know that  $x_n \rightarrow x_0 \in \overline{\Omega}$ , while  $\Omega_n \rightarrow \mathbb{R}^N$  if  $x_0 \in \Omega$  and  $\Omega_n \rightarrow \mathbb{R}_+^N$  if  $x_0 \in \partial\Omega$ . Using the elliptic regularity  $w_n$  is bounded in  $C_{\text{loc}}^{2,\delta}(\overline{\mathbb{R}^N})$ ,  $\delta \in (0, 1)$ . Therefore, passing to the limit through a subsequence and taking into account (3.4) we get a solution  $0 < w \leq 1$  of

$$-\Delta w = \lambda w^p \quad \text{in } \mathbb{R}^N \text{ if } x_0 \in \Omega,$$

or

$$-\Delta w = \lambda w^p \quad \text{in } \mathbb{R}_+^N \text{ if } x_0 \in \partial\Omega,$$

(for the case  $x_0 \in \partial\Omega$  we need to straighten the boundary of  $\Omega$  near  $x_0$  before introducing the scaling, see for instance [10].)

If  $\lambda > 0$  and  $1 < p < (N+2)/(N-2)$ , we arrive at a contradiction as consequence of [10].

■

**Corollary 3.5.** *Assume  $\beta > 1$ ,  $\lambda > 0$  and  $1 < p < (N+2)/(N-2)$ . Then, there exists a priori bound of positive solutions of (1.1).*

*Proof.* Assume that there exists a sequence of positive solutions  $u_n$  such that  $\|u_n\|_\infty \rightarrow \infty$ . Then, by Proposition 3.4 we get

$$u_n \leq \|u_n\|_\infty \leq C \left( \int_\Omega u_n^\beta \right)^{1/p},$$

and by Lemma 3.1

$$\left( \int_\Omega u_n^\beta \right)^{1/p} \leq C.$$

■

The result is also true for any  $\beta < p$ .

**Proposition 3.6.** *Assume that  $0 < \beta < p < (N + 2)/(N - 2)$ ,  $p > 1$ , and  $\lambda \in \Lambda$ , with  $\Lambda \subset \mathbb{R}_+$  compact such that  $0 \notin \Lambda$ . Then, there exists a priori bound of positive solutions of (1.1).*

*Proof.* We use again a Gidas-Spruck argument. With the same notation that Proposition 3.4 we get

$$-\Delta w_n = \lambda w_n^p + M_n^{-p} \int_{\Omega} u_n^{\beta} \quad \text{in } \Omega_n.$$

But,

$$M_n^{-p} \int_{\Omega} u_n^{\beta} \leq M_n^{-p+\beta} |\Omega| \rightarrow 0,$$

and so passing to the limit we again obtain

$$-\Delta w = \lambda w^p \quad \text{in } \mathbb{R}^N \text{ or } \mathbb{R}_+^N.$$

■

Finally, we analyze the case  $p \leq 1$ . Observe that when  $p = 1 > \beta$  we will show that there exists bifurcation from infinity at  $\lambda = \lambda_1 > 0$  (see Theorem 1.1). So, we study the case  $p < 1$  and  $\beta \leq 1$ .

**Proposition 3.7.** *Assume that  $p < 1$  and  $\beta \leq 1$ , and  $\lambda \in \Lambda$ , with  $\Lambda \subset \mathbb{R}_+$  compact such that  $0 \notin \Lambda$ . Then, there exists a priori bound of positive solutions of (1.1).*

*Proof.* Assume that there exists a sequence  $\lambda_n \rightarrow \lambda_0 > 0$  and positive solutions  $u_n$  of (1.1) such that  $\|u_n\|_{\infty} \rightarrow \infty$ . Denote by

$$w_n := \frac{u_n}{\|u_n\|_{\infty}}.$$

It is clear that  $w_n$  verifies

$$-\Delta w_n = \lambda_n w_n^p \|u_n\|_{\infty}^{p-1} + \|u_n\|_{\infty}^{\beta-1} \int_{\Omega} w_n^{\beta} \quad \text{in } \Omega, \quad w_n = 0 \quad \text{on } \partial\Omega.$$

Then,  $w_n \rightarrow w$  in  $C^2(\overline{\Omega})$  being  $w$  a solution of

$$-\Delta w = 0 \quad \text{if } \beta < 1 \quad -\Delta w - \int_{\Omega} w = 0 \quad \text{if } \beta = 1.$$

In the first case, it is clear that  $w \equiv 0$ . In the second one, since there exists positive solution for  $\lambda_n > 0$  we get that  $\sigma_1 > 0$ , and then  $w \equiv 0$ . In both cases, we arrive at contradiction because  $\|w\|_{\infty} = 1$ . ■

In the following result, we show that (1.1) does not possess classical positive solutions for  $\beta = p > (N + 2)/(N - 2)$  and  $\lambda$  large.

**Proposition 3.8.** *Assume that  $\Omega$  is bounded and starshaped with respect to some point  $x_0 \in \Omega$ ,  $\beta = p > (N + 2)/(N - 2)$  and  $\lambda > C(N)$ , for some positive constant depending on  $N$ . Then, (1.1) does not possess positive solution.*



*Proof.* We are going to use a Pohozaev's argument, see for instance Chapter 1.5 in [12]. Multiplying (1.1) by  $x \cdot \nabla u$  we get

$$\frac{N-2}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 x \cdot n = N \frac{\lambda}{p+1} \int_{\Omega} u^{p+1} + N \int_{\Omega} u^{\beta} \int_{\Omega} u,$$

and then

$$0 < \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 x \cdot n = \lambda \left( \frac{N}{p+1} - \frac{N-2}{2} \right) \int_{\Omega} u^{p+1} + \left( \frac{N+2}{2} \right) \int_{\Omega} u^{\beta} \int_{\Omega} u.$$

By Hölder inequality, we get (using  $\beta = p$ )

$$0 < \int_{\Omega} u^{\beta} \int_{\Omega} u \leq C(\Omega) \int_{\Omega} u^{p+1}.$$

Hence

$$0 < \frac{1}{2} \int_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 x \cdot n \leq \left[ \lambda \left( \frac{N}{p+1} - \frac{N-2}{2} \right) + C(\Omega) \frac{N+2}{2} \right] \int_{\Omega} u^{p+1},$$

an absurdum for  $\lambda$  large. ■

With a completely analogous argument, we can prove:

**Corollary 3.9.** *Assume that  $\Omega$  is bounded and starshaped with respect to some point  $x_0 \in \Omega$ ,  $\beta = p < (N+2)/(N-2)$  and  $\lambda < -C(N)$ , for some positive constant depending on  $N$ . Then, (1.1) does not possess positive solution.*

## 4 Proof of the main results

In this section we prove the main results of the paper stated in Section 1. Firstly, observe that if  $(p, \beta) = (1, 1)$  then (1.1) is an eigenvalue problem, and so there exist positive solutions if, and only if,

$$\lambda = \sigma_1.$$

Recall that  $\text{sgn}(\sigma_1) = \text{sgn}(1 - \int_{\Omega} e)$  where  $e$  is defined in (3.1).

So, from now on we assume that  $(p, \beta) \neq (1, 1)$ .

Also, for  $\lambda = 0$  and  $\beta \neq 1$  there exists a unique positive solution

$$u = e \int_{\Omega} u^{\beta} \implies u = e \left( \int_{\Omega} e^{\beta} \right)^{1/(1-\beta)}.$$

(the case  $\beta = 1$  is an eigenvalue problem). Moreover, by Proposition 2.9,  $u$  is stable for  $\beta < 1$  and unstable for  $\beta > 1$ . So, we assume  $\lambda \neq 0$ .

### 4.1 Some useful results

A first attempt to study (1.1) is consider

$$R = \int_{\Omega} u^{\beta},$$

and then we have to study the equation

$$\begin{cases} -\Delta u = \lambda u^p + R & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.1)$$

and after that, to find a point fixed of

$$R = \int_{\Omega} u_R^\beta \iff 1 = \int_{\Omega} w_R^\beta \equiv h(R) \quad (4.2)$$

being  $u_R$  a positive solution of (4.1) and  $w_R = u_R/R^{1/\beta}$  and so positive solution of

$$\begin{cases} -\Delta w = \lambda R^{(p-1)/\beta} w^p + R^{(\beta-1)/\beta} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.3)$$

In the following result we study in detail the map  $R \mapsto h(R)$ .

**Proposition 4.1.** *Assume  $R > 0$ .*

- a) *Assume  $p = 1$ . Then (4.3) possesses a positive solution, denoted by  $w_R$ , if, and only if,  $\lambda < \lambda_1$ . The solution is unique. Moreover,*

$$w_R = e_\lambda R^{\frac{\beta-1}{\beta}}, \quad (4.4)$$

*being  $e_\lambda$  the unique positive solution of*

$$\begin{cases} (-\Delta - \lambda)e_\lambda = 1 & \text{in } \Omega, \\ e_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5)$$

- b) *Assume  $p < 1$ . Then (4.3) possesses a unique positive solution, denoted by  $w_R$ , for all  $\lambda \in \mathbb{R}$ . Moreover, the map  $R \in (0, \infty) \mapsto h(R)$  is continuous and derivable. For  $\lambda > 0$*

$$\lim_{R \rightarrow 0} h(R) = \infty.$$

- (a) *When  $\beta = 1$ .*

- i. *If  $\lambda > 0$ ,  $R \mapsto h(R)$  is decreasing and*

$$\lim_{R \rightarrow \infty} h(R) = \int_{\Omega} e.$$

- ii. *If  $\lambda < 0$ ,  $R \mapsto h(R)$  is increasing and*

$$\lim_{R \rightarrow 0} h(R) = 0, \quad \lim_{R \rightarrow \infty} h(R) = \int_{\Omega} e.$$

- (b) *When  $\beta > 1$ ,*

- i. *If  $\lambda > 0$*

$$\lim_{R \rightarrow \infty} h(R) = \infty.$$

ii. If  $\lambda < 0$ ,  $R \mapsto h(R)$  is increasing and

$$\lim_{R \rightarrow 0} h(R) = 0, \quad \lim_{R \rightarrow \infty} h(R) = \infty.$$

(c) When  $\beta < 1$  and  $\lambda > 0$ ,  $R \mapsto h(R)$  is decreasing, and for all  $\lambda$

$$\lim_{R \rightarrow \infty} h(R) = 0.$$

Moreover,

i. If  $\beta < p$  and  $\lambda < 0$ ,  $R \mapsto h(R)$  is decreasing, and for all  $\lambda$

$$\lim_{R \rightarrow 0} h(R) = \infty.$$

ii. If  $\beta = p$ , the map  $R \mapsto h(R)$  is decreasing and

$$\lim_{R \rightarrow 0} h(R) = \begin{cases} \infty & \text{if } \lambda > 0, \\ \rho_0(\lambda) & \text{if } \lambda < 0, \end{cases}$$

where

$$\rho_0(\lambda) \in \left( (-1/\lambda) \int_{\Omega} \varphi_1^{\beta}, (-1/\lambda) |\Omega| \right), \quad (4.6)$$

$\varphi_1$  is the positive eigenfunction associated to  $\lambda_1$  such that  $\|\varphi_1\|_{\infty} = 1$ , and  $\rho_0(\lambda)$  is a non-decreasing function in  $\lambda$  for  $\lambda < 0$ .

iii. If  $\beta > p$ ,

$$\lim_{R \rightarrow 0} h(R) = \begin{cases} \infty & \text{if } \lambda > 0, \\ 0 & \text{if } \lambda < 0. \end{cases}$$

c) Assume  $p > 1$  and  $\lambda < 0$ , then there exists a unique positive solution, denoted by  $w_R$ , of (4.3). Moreover, the map  $R \in (0, \infty) \mapsto h(R)$  is continuous and derivable.

(a) When  $\beta = 1$ . The map  $R \mapsto h(R)$  is decreasing and

$$\lim_{R \rightarrow 0} h(R) = \int_{\Omega} e, \quad \lim_{R \rightarrow \infty} h(R) = 0.$$

(b) When  $\beta > 1$ .

i. If  $\beta > p$ ,  $R \mapsto h(R)$  is increasing and

$$\lim_{R \rightarrow 0} h(R) = 0, \quad \lim_{R \rightarrow \infty} h(R) = +\infty.$$

ii. If  $\beta = p$ , the map  $R \mapsto h(R)$  is increasing

$$\lim_{R \rightarrow 0} h(R) = 0, \quad \lim_{R \rightarrow \infty} h(R) = \rho_0(\lambda),$$

with  $\rho_0(\lambda)$  as in (4.6).

iii. If  $\beta < p$

$$\lim_{R \rightarrow 0} h(R) = 0, \quad \lim_{R \rightarrow \infty} h(R) = 0.$$

(c) When  $\beta < 1$ . The map  $R \mapsto h(R)$  is decreasing and

$$\lim_{R \rightarrow 0} h(R) = +\infty \quad \lim_{R \rightarrow \infty} h(R) = 0.$$

*Proof.* a) Assume that  $p = 1$ , then  $w_R$  verifies

$$(-\Delta - \lambda)w_R = R^{(\beta-1)/\beta}$$

and the result of paragraph a) is obtained easily.

For the other cases, it is clear that  $(\underline{w}, \overline{w}) = (0, Ke)$  is a pair of sub-supersolution of (4.3) for  $K$  verifying

$$K \geq \lambda K^p e^p R^{(p-1)/\beta} + R^{(\beta-1)/\beta}. \quad (4.7)$$

It is enough to take  $K$  large in any case.

On the other hand, for  $\lambda \geq 0$  and  $p < 1$  the uniqueness follows by [3] and for  $\lambda < 0$  thanks to  $\lambda R^{(p-1)/\beta} w^p$  is a decreasing map in  $R$ .

The continuity and derivability of the map  $R \mapsto w_R$  is standard.

If  $\lambda \geq 0$  we have that  $-\Delta w \geq R^{(\beta-1)/\beta}$  and so

$$w_R \geq R^{(\beta-1)/\beta} e. \quad (4.8)$$

Analogously, if  $\lambda \leq 0$  we have that

$$w_R \leq R^{(\beta-1)/\beta} e. \quad (4.9)$$

Moreover, if  $\lambda \geq 0$  and  $p < 1$  we have that  $-\Delta w \geq \lambda w^p R^{(p-1)/\beta}$  and then

$$w_R \geq R^{-1/\beta} \lambda^{1/(1-p)} w_1, \quad (4.10)$$

where  $w_1$  is the unique positive solution of (2.6).

Also, by the maximum principle for  $\lambda < 0$  we get

$$\|w_R\|_\infty^p \leq \frac{R^{(\beta-p)/\beta}}{-\lambda}. \quad (4.11)$$

Finally,  $\varepsilon \varphi_1$  is subsolution of (4.3),  $\|\varphi_1\|_\infty = 1$ , if

$$\varepsilon \lambda_1 \leq \lambda \varepsilon^p \varphi_1^p R^{(p-1)/\beta} + R^{(\beta-1)/\beta}. \quad (4.12)$$

b) Assume that  $p < 1$ . Then, it is clear by (4.10) that for  $\lambda > 0$

$$\lim_{R \rightarrow 0} h(R) = +\infty,$$

and for  $\beta > 1$  by (4.8)

$$\lim_{R \rightarrow \infty} h(R) = +\infty.$$

Take now  $\lambda \leq 0$ , then it is clear by (4.9) that

$$\lim_{R \rightarrow 0} h(R) = 0 \quad \text{if } \beta > 1, \quad \lim_{R \rightarrow \infty} h(R) = 0 \quad \text{if } \beta < 1,$$

and by (4.11)

$$\lim_{R \rightarrow 0} h(R) = 0 \quad \text{if } \beta > p, \quad \lim_{R \rightarrow \infty} h(R) = 0 \quad \text{if } \beta < p.$$

Now, consider  $\beta > 1$  and  $\lambda < 0$ . Observe that the map  $R \mapsto \lambda R^{(p-1)/\beta} + R^{(\beta-1)/\beta}$  is increasing, and so  $R \mapsto w_R$  also. In this case, for (4.12) is enough

$$\varepsilon \lambda_1 - \lambda \varepsilon^p R^{(p-1)/\beta} = R^{(\beta-1)/\beta}. \quad (4.13)$$

From this equality we deduce that  $\varepsilon(R) \rightarrow \infty$  as  $R \rightarrow \infty$ , and then  $h(R) \rightarrow \infty$ .

For  $\beta < 1$  we have to distinguish several cases. If  $\beta < p$  and  $\lambda < 0$  then again by (4.13), we get that  $\varepsilon(R) \rightarrow \infty$  as  $R \rightarrow 0$ . For the case  $\beta = p$  we have that

$$\varepsilon^p(R) \rightarrow -\frac{1}{\lambda} \quad \text{as } R \rightarrow 0.$$

Moreover, by (4.11) we deduce that  $h(R) \leq \frac{-1}{\lambda} |\Omega|$ .

For  $\lambda > 0$ , for (4.7) is enough

$$K - \lambda \|e\|_\infty^p K^p R^{(p-1)/\beta} = R^{(\beta-1)/\beta}. \quad (4.14)$$

If  $p, \beta < 1$  it is clear that  $K(R) \rightarrow 0$  as  $R \rightarrow \infty$ .

Finally, for  $\beta = 1$  and  $\lambda > 0$  observe that

$$e \leq w_R \leq K(R)e$$

and  $K(R) \rightarrow 1$  as  $R \rightarrow \infty$  by (4.14).

For  $\lambda < 0$ ,  $w_R \leq e$  and  $\varepsilon(R)e$  is subsolution if

$$\varepsilon - \lambda \varepsilon^p C R^{p-1} = 1. \quad (4.15)$$

It is clear that  $\varepsilon(R) \rightarrow 1$  as  $R \rightarrow \infty$ .

Observe also that in the particular case  $\beta = p$ ,  $w_R$  verifies

$$-\Delta w = R^{(\beta-1)/\beta} (\lambda w^p + 1)$$

and then, by the maximum principle

$$\lambda w^p(x) + 1 \geq 0 \quad \text{for all } x \in \Omega. \quad (4.16)$$

Indeed, if  $\lambda \geq 0$  then (4.16) is clear. If  $\lambda < 0$  observe that

$$\lambda w^p(x) + 1 \geq \lambda \|w\|_\infty^p + 1 \geq 0.$$

Then, if  $R_1 < R_2$  and  $\beta < 1$ , we get that  $w_{R_2}$  is sub-solution of (4.3) for  $R = R_1$ , and then  $w_{R_2} < w_{R_1}$ . This proves that  $R \mapsto h(R)$  is decreasing.

Hence, there exists the following limit

$$\lim_{R \downarrow 0} h(R) := \rho_0(\lambda).$$

Moreover, if  $\lambda_1 < \lambda_2 < 0$ ,  $w_{\lambda_1, R}$  is subsolution of the equation (4.3) with  $\lambda = \lambda_2$ , and so  $w_{\lambda_1, R} < w_{\lambda_2, R}$ . Taking limit we have that

$$\rho_0(\lambda_1) \leq \rho_0(\lambda_2).$$

Finally, assume that  $\beta \leq p$ . Observe that  $w_R$  verifies

$$-\Delta w = R^{(\beta-1)/\beta}(\lambda R^{(p-\beta)/\beta} w^p + 1).$$

Observe that by the maximum principle and since  $\lambda < 0$  we get

$$\lambda R^{(p-\beta)/\beta} w^p(x) + 1 \geq \lambda R^{(p-\beta)/\beta} \|w\|_\infty^p(x) + 1 \geq 0.$$

Take  $R_1 < R_2$ , then

$$R_1^{(\beta-1)/\beta}(\lambda R_1^{(p-\beta)/\beta} w_{R_1}^p + 1) \geq R_2^{(\beta-1)/\beta}(\lambda R_2^{(p-\beta)/\beta} w_{R_1}^p + 1),$$

and then  $w_{R_1}$  is a supersolution of the equation (4.3) with  $R = R_2$ . We conclude that  $w_{R_1} \geq w_{R_2}$ .

c) Assume that  $p > 1$  and  $\lambda < 0$ . From (4.9) we have that  $h(R) \rightarrow 0$  as  $R \rightarrow 0$  if  $\beta > 1$  and  $h(R) \rightarrow 0$  as  $R \rightarrow \infty$  if  $\beta < 1$ .

In this case if  $\beta < 1$  it is clear that  $\varepsilon(R) \rightarrow \infty$  as  $R \rightarrow 0$  from (4.13).

Now, assume  $\beta > 1$ . If  $p < \beta$  then  $\varepsilon(R) \rightarrow \infty$  as  $R \rightarrow \infty$ , if  $p = \beta$ ,  $\varepsilon^p(R) \rightarrow -1/\lambda$  and for  $p > \beta$  we have that  $K(R) \rightarrow 0$ .

Finally, for  $\beta = 1$ , and using again (4.15), we have that  $h(R) \rightarrow \int_\Omega e$  if  $R \rightarrow 0$ . ■

In the following result we prove a stability result of a positive solution  $u_0$  of (1.1), obtained such that  $u_0 = u_{R_0}$  for some  $R_0 > 0$ , in function on the map  $h$  defined in (4.2).

**Proposition 4.2.** *Let  $u_0$  be a positive solution of (1.1) obtained such that  $u_0 = u_{R_0}$  for some  $R_0 > 0$ . Then, if*

$$h'(R_0) < 0 \quad (\text{resp. } h'(R_0) > 0) \quad \text{then } u_0 \text{ is stable (resp. } u_0 \text{ is unstable)}.$$

*Proof.* Let  $u_0 = u_{R_0}$  a positive solution of (1.1). Assume that  $h'(R_0) < 0$ , we want to show that

$$\lambda_1(-\Delta - \lambda p u_0^{p-1}; \beta; u_0^{\beta-1}) > 0, \tag{4.17}$$

(analogous argument in the case  $h'(R_0) > 0$ ).

First, observe that the map  $R \mapsto u_R$  is increasing ( $u_R$  defined in (4.1)), and so its derivative  $u'_R > 0$  in  $\Omega$ , being  $u'_R$  the unique solution of

$$-\Delta u'_R = \lambda p u_R^{p-1} u'_R + 1 \quad \text{in } \Omega, \quad u'_R = 0 \quad \text{on } \partial\Omega.$$

On the other hand, observe that since  $h'(R_0) < 0$  and using that  $h(R) = (1/R) \int_\Omega u_R^\beta$ , we get

$$\beta \int_\Omega u_{R_0}^{\beta-1} u'_{R_0} < \frac{\int_\Omega u_{R_0}^\beta}{R_0} = h(R_0) = 1.$$

To prove (4.17) we use Proposition 2.2 with  $\bar{u} = u'_{R_0} > 0$ . Indeed, observe that

$$-\Delta u'_{R_0} - \lambda p u_0^{p-1} u'_{R_0} - \beta \int_\Omega u_0^{\beta-1} u'_{R_0} = 1 - \beta \int_\Omega u_0^{\beta-1} u'_{R_0} > 0,$$

and then the stability follows. ■

For the case  $p > 1$  and  $\lambda > 0$  we work with the original equation. In fact, assume  $\beta \geq 1$  and consider the following auxiliari problem:

$$\begin{cases} -\Delta u = \mu u + \lambda u^p + \int_{\Omega} u^{\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.18)$$

**Lemma 4.3.** *Assume  $p > 1$ ,  $\beta \geq 1$  and  $\lambda > 0$ . Then, for  $\mu = \lambda_1$  when  $\beta > 1$  and  $\mu = \sigma_1$  when  $\beta = 1$  bifurcates from the trivial solution a non-bounded continuum  $\mathcal{C}$  of positive solutions of (4.18). Moreover, assuming the existence of a priori bound of (4.18) for  $\mu \in \Lambda$ ,  $\Lambda$  a compact subset of  $\mathbb{R}$ , there exists a positive solution if, and only if,  $\mu < \lambda_1$  if  $\beta > 1$  and  $\mu < \sigma_1$  for  $\beta = 1$ .*

*Proof.* First, observe that if  $u$  is a positive solution of (4.18) we have

$$\mu = \lambda_1(-\Delta - \lambda u^{p-1}; 1; u^{\beta-1}) < \lambda_1(-\Delta; 1; u^{\beta-1}),$$

that is,  $\mu < \lambda_1$  if  $\beta > 1$  and  $\mu < \sigma_1$  in the case  $\beta = 1$ .

That  $\mu = \lambda_1$  for  $\beta > 1$  and  $\mu = \sigma_1$  for  $\beta = 1$  is a bifurcation point from the trivial solution is consequence of the Crandall-Rabinowitz Theorem [5], see also [6].

The existence of an unbounded continuum  $\mathcal{C}$  follows by the classical Rabinowitz Theorem [13]. ■

## 4.2 Proof of Theorem 1.1

Assume that  $p = 1$ . It is clear that (1.1) does not possess positive solution for  $\lambda \geq \lambda_1$ . By Proposition 4.1 a) there exists a unique positive solution for  $\lambda < \lambda_1$ . The stability results follow by Proposition 2.9 a) and b).

We study now the behaviour with respect to  $\lambda$ . Observe that if  $u$  is a positive solution of (1.1) we have

$$(-\Delta - \lambda)u = \int_{\Omega} u^{\beta},$$

and so,

$$u = e_{\lambda} \left( \int_{\Omega} e_{\lambda}^{\beta} \right)^{1/(1-\beta)},$$

and so taking into account that for  $\varphi_1 > 0$  with  $\|\varphi_1\|_{\infty} = 1$ ,  $\varphi_1$  eigenfunction associated to  $\lambda_1$ ,

$$\frac{1}{\lambda_1 - \lambda} \varphi_1 \leq e_{\lambda} \quad \text{in } \Omega,$$

we get that for  $\beta < 1$ ,

$$u \geq \frac{1}{(\lambda_1 - \lambda)^{1/(1-\beta)}} \varphi_1 \left( \int_{\Omega} \varphi_1^{\beta} \right)^{1/(1-\beta)}$$

and so  $\|u\|_{\infty} \rightarrow \infty$  as  $\lambda \rightarrow \lambda_1$ .

Assume that  $\beta > 1$  and consider a sequence  $\lambda_n < \lambda_1$ ,  $\lambda_n \rightarrow \lambda_1$  and  $u_n$  the positive solution of (4.18) for  $\lambda = \lambda_n$ . We know by Proposition 3.3 that  $\|u_n\|_{\infty}$  is bounded, and so passing to the limit we get that  $u_n \rightarrow u_0$  in  $C^2(\bar{\Omega})$  as  $\lambda_n \rightarrow \lambda_1$ , with  $u_0$  positive solution for  $\lambda = \lambda_1$ . Then,  $u_0 \equiv 0$ .

Finally, the behaviour as  $\lambda \rightarrow -\infty$  follows by Lemma 2.8.

### 4.3 Proof of Theorem 1.2

a) Assume  $p < 1 = \beta$ . Assume that  $\sigma_1 > 0$ , then it is clear that  $\lambda > 0$ . Observe that since  $\sigma_1 > 0$ , applying Lemma 2.3 with  $a \equiv b \equiv 1$  and  $m \equiv 0$  we get that  $\int_{\Omega} e < 1$ . Now, the existence and uniqueness follow by Proposition 4.1 b). The stability follows by Proposition 2.9. Finally, observe that (see (2.7))

$$u_{\lambda} = \lambda^{1/(1-p)} u_1,$$

being  $u_1$  the unique solution of (1.1) for  $\lambda = 1$ . From here, we can deduce the behaviour as  $\lambda \rightarrow 0$  and  $\lambda \rightarrow \infty$ . The other cases can be treated similarly.

b) Assume  $p < 1 < \beta$ . The existence and uniqueness in the case  $\lambda < 0$  follow by Proposition 4.1. Also, the stability follows by Proposition 2.9. Now consider  $\lambda > 0$ . Denote

$$e_R := R^{(\beta-1)/\beta} e.$$

Take  $R_0 > 0$  small such that

$$\int_{\Omega} e_{R_0}^{\beta} = R_0^{\beta-1} \int_{\Omega} e^{\beta} < 1.$$

Fix such  $R_0 > 0$ . Now, it is clear  $w_{R_0} \rightarrow e_{R_0}$  in  $L^{\infty}(\Omega)$  as  $\lambda \rightarrow 0$ , hence  $h(R_0) < 1$  for  $\lambda \leq \lambda_0$ , with  $\lambda_0$  small. So, since

$$\lim_{R \rightarrow 0} h(R) = \lim_{R \rightarrow \infty} h(R) = +\infty,$$

there exist at least two positive values  $R_0^1 < R_0 < R_0^2$  such that  $h(R_0^i) = 1$ ,  $i = 1, 2$ , and so two positive solutions  $u_i^{\lambda} = u_{R_0^i}$  of (1.1) for  $\lambda \leq \lambda_0$  with  $u_1 < u_2$ , and

$$h'(R_0^1) < 0 < h'(R_0^2).$$

Thank to Proposition 4.2 we have that  $u_1^{\lambda}$  is stable and  $u_2^{\lambda}$  unstable.

Now, we show that there does not exist positive solutions of (1.1) for  $\lambda$  large. Observe that

$$u \geq \lambda^{1/(1-p)} w_1, \tag{4.19}$$

where  $w_1$  is defined in (2.6), and then

$$-\Delta u \geq \lambda^{1/(1-p)} w_1^p + \lambda^{(\beta-1)/(1-p)} \int_{\Omega} w_1^{\beta-1} u$$

and so

$$\lambda_1(-\Delta; \lambda^{(\beta-1)/(1-p)}; w_1^{\beta-1}) > 0.$$

This is an absurdum because by Lemma 2.6

$$\lambda_1(-\Delta; \lambda^{(\beta-1)/(1-p)}; w_1^{\beta-1}) \rightarrow -\infty \quad \text{as } \lambda \rightarrow \infty.$$

Then, we can define

$$\Lambda := \{\lambda \in \mathbb{R} : \text{there exists at least a positive solution of (1.1)}\}.$$



We have proved that  $0 < \bar{\lambda} := \sup \Lambda < \infty$ . Thanks to the bounds by Proposition 3.3, there exists positive solution for  $\lambda = \bar{\lambda}$ . Now, it is clear that if  $\lambda \in (0, \bar{\lambda})$  then  $(\varepsilon w_1, u_{\bar{\lambda}})$  is a pair of sub-supersolution of (1.1) with  $\varepsilon$  small and  $u_{\bar{\lambda}}$  a positive solution of (1.1) for  $\lambda = \bar{\lambda}$ . Observe that this method works for non-local equation, see for instance [9].

On the other hand, consider  $u_1^\lambda$  for  $\lambda \in (0, \lambda_0)$ . Since for  $\lambda = 0$  the solution is unstable, we can assure that

$$\lim_{\lambda \rightarrow 0} \|u_1^\lambda\|_\infty = 1.$$

Finally, the behaviour of the solution as  $\lambda \rightarrow -\infty$  follows by Lemma 2.8.

c) Assume  $p, \beta < 1$ . In this case, it is clear by Propositions 4.1 and 4.2 the existence, uniqueness and stability for  $\lambda > 0$ .

(a) Suppose that  $\beta < p$ . In this case we have again by Proposition 4.1 the existence and uniqueness for all  $\lambda \in \mathbb{R}$  and the stability follows by Proposition 2.9.

(b) Suppose  $\beta = p$ . Observe that since the map  $h(R)$  is decreasing, in case of existence of positive solution, it is unique. Moreover, by (4.6) and since  $\rho_0(\lambda)$  is non-decreasing, there exists a unique value  $\lambda_0 < 0$  such that

$$\rho_0(\lambda) \leq 1 \quad \text{for } \lambda \leq \lambda_0 \text{ and } \rho_0(\lambda) > 1 \quad \text{for } \lambda \in (\lambda_0, 0).$$

Hence, there exists a positive solution of (1.1) if, and only if,  $\lambda > \lambda_0$ . Then,

$$\lim_{R \rightarrow 0} h(R) \leq 1 \quad \text{for } \lambda < \lambda_0, \quad \lim_{R \rightarrow 0} h(R) > 1 \quad \text{for } \lambda > \lambda_0,$$

and

$$\lim_{\lambda \rightarrow \lambda_0} \|u_\lambda\|_\infty = 0.$$

Again, thanks to Proposition 4.2 we know that the solution is stable.

(c) Suppose  $\beta > p$ . With a similar argument to the used in the paragraph b) we can show the existence of two positive solutions for  $\lambda$  negative and small. Indeed, in this case

$$\lim_{R \rightarrow 0} h(R) = \lim_{R \rightarrow \infty} h(R) = 0,$$

and there exist at least two positive values  $R_0^1 < R_0 < R_0^2$  such that  $h(R_0^i) = 1$ ,  $i = 1, 2$ , and so two positive solutions  $u_i^\lambda = u_{R_0^i}$  of (1.1) for  $\lambda \leq \lambda_0$  with  $u_1^\lambda < u_2^\lambda$ , and

$$h'(R_0^1) > 0 > h'(R_0^2),$$

and then  $u_1^\lambda$  is unstable and  $u_2^\lambda$  stable.

We prove now the non-existence of positive solutions for  $\lambda$  very negative. Indeed, observe that by (2.5) we have

$$-\lambda \leq C \|u\|_\infty^{\beta-p} \leq C \tag{4.20}$$

this last inequality by Lemma 3.1. Then, if there exists a positive solution of (1.1) we get that  $\lambda \geq -C$ .

Again, we can define

$$\Lambda := \{\lambda \in \mathbb{R} : \text{there exists at least a positive solution of (1.1)}\}.$$

We know that  $\underline{\lambda} := \inf \Lambda > -\infty$  and  $\underline{\lambda} < 0$ , and using as sub-supersolution the pair  $(u_{\underline{\lambda}}, Ke)$  for  $K$  large, we prove the existence of positive solution for all  $\lambda \in (\underline{\lambda}, 0)$ . Finally, by (4.20) it can not occur that for a sequence  $(\lambda_n, u_n)$  we have  $\lambda_n \rightarrow \lambda_0 < 0$  and  $\|u_n\|_\infty \rightarrow 0$ . Moreover, since the solution for  $\lambda = 0$  we can conclude that

$$\lim_{\lambda \rightarrow 0} \|u_1^\lambda\|_\infty = 0.$$

#### 4.4 Proof of Theorem 1.3

a) Assume  $\beta = 1 < p$ . If  $\sigma_1 > 0$  and there exists a positive solution of (1.1) then  $\lambda > 0$ . Now, suppose that  $\lambda > 0$ . Then, thanks to that  $1 < p < (N+2)/(N-2)$  we have a priori bounds for the positive solutions of (4.18) by Proposition 3.6, and then applying Lemma 4.3 there exists at least a positive solution  $u_\mu$  for all  $\mu < \sigma_1$ . Since  $\sigma_1 > 0$ , there exists a positive solution  $u_0$  for  $\mu = 0$ , that is,  $u_0$  is solution of (1.1). Finally, by (2.7) we conclude the behaviour as  $\lambda \rightarrow 0$  and  $\lambda \rightarrow +\infty$ .

If  $\sigma_1 = 0$  then  $\lambda = 0$ , and so there exist infinite positive solutions of (1.1).

Finally, if  $\sigma_1 < 0$  then  $\lambda < 0$ , and by Lemma 2.3 we have that

$$\int_{\Omega} e > 1,$$

and then the result follows by Proposition 4.1. By Lemma 2.8 we know the limit as  $\lambda \rightarrow -\infty$ .

Finally, again it can be proved that  $u_\lambda = (-\lambda)^{\frac{1}{1-p}} u_{-1}$  being  $u_{-1}$  a positive solution of (1.1) for  $\lambda = -1$ . We conclude (1.7).

b) Assume that  $p, \beta > 1$ . If  $\beta > p$  there exists a unique and unstable positive solution for  $\lambda \leq 0$  by Proposition 4.1.

Consider now the case  $\beta = p$ . In this case the map  $h(R)$  is increasing,  $\lim_{R \rightarrow 0} h(R) = 0$  and  $\lim_{R \rightarrow \infty} h(R) = \rho_0(\lambda)$ . Moreover, by (4.6) and since  $\rho_0(\lambda)$  is non-decreasing, there exists a unique value  $\lambda_0 < 0$  such that

$$\rho_0(\lambda) \leq 1 \quad \text{for } \lambda \leq \lambda_0 \quad \text{and} \quad \rho_0(\lambda) > 1 \quad \text{for } \lambda \in (\lambda_0, 0).$$

Hence, there exists a positive solution of (1.1) if, and only if,  $\lambda > \lambda_0$ . The solution is unique and, by Proposition 4.2, it is unstable. Finally, we get that

$$\lim_{\lambda \rightarrow \lambda_0} \|u_\lambda\|_\infty = +\infty.$$

For the case  $\beta < p$ , first we show that  $w_R \rightarrow R^{(\beta-1)/\beta} e$  as  $\lambda \uparrow 0$ . Indeed, we can prove that

$$w_R \leq K(R)e \quad \text{for } \lambda < 0,$$

for some constant  $K(R) > 0$  independent of  $\lambda$ . Take now  $R_0 > 0$  such that

$$R_0^{(\beta-1)} \int_{\Omega} e^\beta > 1$$

and then for  $\lambda$  is small,  $h(R_0) > 1$ . Hence, since

$$\lim_{R \rightarrow 0} h(R) = \lim_{R \rightarrow \infty} h(R) = 0,$$

there exist two values  $R_0^1 < R_0 < R_0^2$  such that

$$h(R_0^1) = 1, \quad h'(R_0^2) < 0 < h'(R_0^1).$$

This proves the existence of two positive solutions  $u_1^\lambda < u_2^\lambda$  for  $\lambda$  small and negative, with

$$u_i^\lambda = u_{R_i}, \quad R_0^1 < R_0^2,$$

and  $u_1^\lambda$  unstable and  $u_2^\lambda$  stable. Observe also that since there exist a priori bounds for  $\lambda < 0$  and for  $\lambda = 0$  the solution is unstable, the sequence  $\|u_2^\lambda\|_\infty \rightarrow \infty$  as  $\lambda \rightarrow 0$ .

Also, observe that by (2.5) we get  $-\lambda \leq \|u\|^{\beta-p}|\Omega|$  and then  $\|u\|_\infty \rightarrow 0$  as  $\lambda \rightarrow -\infty$  and then  $\int_\Omega u^\beta \rightarrow 0$ , a contradiction with Lema 3.1. That is, there does not exist positive solution of (1.1) for  $\lambda$  very negative. Hence, we can define the set

$$\Lambda := \{\lambda \in \mathbb{R} : \text{there exists at least a positive solution of (1.1)}\}.$$

We have proved that  $-\infty < \inf \Lambda := \bar{\lambda} < 0$  and it can shown similarly to the other cases the existence of solution for all  $\lambda \in [\bar{\lambda}, 0)$ .

Lemma 2.8 provides us with the behaviour as  $\lambda \rightarrow -\infty$  and  $\lambda \rightarrow \infty$ .

In all the cases, when  $\beta, p > 1$ ,  $\lambda > 0$  and assuming a priori bounds, we have solutions of (4.18) for all  $\mu < \lambda_1$ , and then for  $\mu = 0$ .

c) Assume now that  $\beta < 1 < p$ . Again, the existence, uniqueness and stability of positive solution for  $\lambda \leq 0$  follow by Proposition 4.1. We prove the non-existence for  $\lambda$  large. Observe that for  $\lambda > 0$  we have by Lemma 3.1 that

$$u \geq Ke,$$

for

$$K = \left( \int_\Omega e^\beta \right)^{1/(1-\beta)}.$$

Then,

$$-\Delta u \geq \lambda(Ke)^{p-1}u + \int_\Omega u^\beta,$$

and hence

$$\lambda_1(-\Delta - \lambda(Ke)^{p-1}; 0; 0) > 0,$$

an absurdum for  $\lambda$  large.

We can define

$$\Lambda := \{\lambda \in \mathbb{R} : \text{there exists at least a positive solution of (1.1)}\}.$$

We know that  $\bar{\lambda} := \sup \Lambda < \infty$ . Moreover, since there exists a unique and stable positive solution for  $\lambda = 0$ , say  $u_0$ , using again Proposition 20.6 in [2] we can conclude that in a neighborhood  $\mathcal{N} \subset \mathbb{R} \times L^\infty(\Omega)$  of  $(\lambda, u) = (0, u_0)$  there exists a unique stable positive solution. Then,  $\bar{\lambda} > 0$ . Again, using the sub-supersolution method with  $(\underline{u}, \bar{u}) = (\varepsilon\varphi_1, u_{\bar{\lambda}}^-)$  and  $\varepsilon > 0$  small, it can be proven that there exists at least a positive solution for all  $\lambda \leq \bar{\lambda}$ .

Finally, since for  $\lambda = 0$  there exists a unique positive solution of (1.1),  $u_0$ , and it is stable, by Theorem 17.1 in [2], then there exists an unbounded subcontinuum  $\Sigma$  containing  $(0, u_0)$ . Fix our attention in the case  $\lambda$  positive. Since there does not exist positive solution for  $\lambda$  large, we get that  $\text{Proj}_{\mathbb{R}}(\Sigma)$  is bounded. On the other hand, observe that since for  $\lambda \in (0, \lambda_0)$ , for  $\lambda_0$  small, we have stable positive solutions  $u_\lambda$ , for all  $\lambda \in (0, \lambda_0)$  there exists at least another positive solution  $w_\lambda \in \Sigma$  with  $w_\lambda \neq u_\lambda$ . Moreover, since  $\Sigma$  is unbounded, and for  $\lambda = 0$  the solution is stable, we have that  $\|w_\lambda\|_\infty \rightarrow +\infty$  as  $\lambda \downarrow 0$ .

**Acknowledgements.** The authors thank to Prof. C. O. Alves for his helpful comments.

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